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LARGE TIME SOLUTIONS FOR TEMPERATURES IN A SEMI-INFINITE BODY WITH A DISK HEAT SOURCE

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Abstract – This paper presents a series solution for the local surface temperature history for a semi-infinite body heated only over a circular region. Inside this region the heat flux is constant with time and position while outside the circular area the surface is insulated. A number of approximate solutions are available in the literature. One exact solution is available but it is in the form of an integral with an infinite domain. The solution developed herein is much more convenient to use for all dimensionless times except the smallest. Extensive curves and tables are provided also.

In addition to the surface solution there is a solution for certain interior locations also for 'large' times. The solution is important because it is a basic geometry in heat conduction and is frequently needed in connection with cylindrical bodies. The solution can be utilized as a building block for related finite geometries for time-variable heating and for symmetric spatially-varying heat flux cases. It can also be used in a promising new calculation method that is called the surface element method.

NOMENCLATURE

a,	radius of heated area;
c,	radius of region for average temperature;
C_k,	coefficient defined by equation (19);
c_p,	specific heat;
$D_i(r)$,	function defined by equation (14a);
$E(\cdot)$,	complete elliptic integral of the second kind;
I_2,	function given by equation (18);
k,	thermal conductivity;
$K(\cdot)$,	complete elliptic integral of the first kind;
$J_i(\cdot)$,	Bessel function of the first kind;
q,	heat flux;
r,	radial coordinate;
t,	time;
T,	temperature;
U_{kj},	function given by equation (20);
$V_i(x)$,	function given by equation (38a);
$W_j(x)$,	function given by equation (38b);
z,	axial coordinate;
α,	thermal diffusivity;
$\Gamma(n)$,	Gamma function;
$\Gamma(n, x)$,	incomplete Gamma function;
ρ,	density.

INTRODUCTION

THE CASE of a semi-infinite solid heated by a disk heat source is a basic building block in transient heat conduction. Though many analyses have been made no exact solution has been previously developed that is valid for all times and locations within the body and that can be conveniently evaluated. Only for the centerline is there a convenient exact transient solution available. This paper provides an exact series solution for any surface location and for all times except for the smallest. Another solution is included that is valid for some interior locations and for 'large' times.

Some of the pioneering solutions were employed for analysis of electric contacts and sliding contacts; see the book by Holm and Holm [1]. Oosterkamp [2] was interested in heat dissipation at the anode of an X-ray tube. More recently Yovanovich and co-workers [3–5] presented results motivated by the contact conductance problem. None of these papers provided an exact solution for the transient temperature distribution. An exact solution is known, however; it is in Carslaw and Jaeger [6] and is attributed to Oosterkamp. It provided a starting point for some of the work in this paper. It is in the form of an integral with limits of zero and infinity and has an integrand involving error and Bessel functions. The integral is difficult to accurately evaluate using a numerical procedure because the domain is infinite and because of the sinusoidal nature of the Bessel functions. Though this integral presents a solution valid for any radius and depth, accurate values are difficult to obtain except along the centerline.

Thomas [7] gave an exact solution in terms of tabulated functions for the *steady state* at the heated surface. Analytical expressions were presented by Beck [8] for the average transient temperature over the radial region from the center to the heated region radius. Any depth in the body was considered. Closed form series expressions were given and some extremely accurate approximations were provided. Local temperatures at any position in the body were not covered, however.

Two of the main purposes of this paper are to provide series expression valid for (1) any radial location at the heated surface and (2) certain internal locations. Solutions are derived for 'large' times which also converge for relatively small times such as $\alpha t/a^2 \approx$

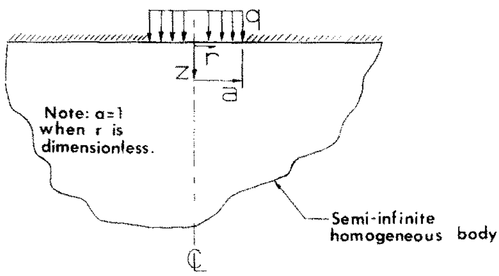

FIG. 1. Semi-infinite body heated over a disk-shaped region centered at $r = 0$ and $z = 0$ and insulated elsewhere at $z = 0$.

0.04 for $r < a$ where a is the radius of the disk source. A solution is also presented for the average temperature from the centerline to any radial location. It is further shown how the solution can be used to generate a solution for an arbitrary-in-r heat flux.

GEOMETRY AND MATHEMATICAL DESCRIPTION

The geometry and coordinates are shown in Fig. 1. The body is isotropic, homogeneous and semi-infinite; it is conveniently described using the cylindrical coordinates r and z. The surface is insulated except over the circular region from $r = 0$ to a where there is a constant heat flux q.

A mathematical statement of the problem is the solution of

$$k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}\right] = \rho c_p \frac{\partial T}{\partial t} \tag{1a}$$

$$-k\frac{\partial T(r,0,t)}{\partial z} = \begin{cases} q \text{ for } 0 < r < a \\ 0 \text{ for } r > a \end{cases} \tag{1b}$$

$$T(r, z, t) \to 0 \text{ for } r \to \infty \text{ and } z \to \infty \tag{1c}$$

$$T(r, z, 0) = 0. \tag{1d}$$

With the initial temperature being zero, the symbol T can be interpreted as the temperature rise. The properties k, ρ and c_p are assumed to be independent of temperature and position.

AVAILABLE SOLUTIONS

The exact solution given by Carslaw and Jaeger ([6], p. 264) is

$$\frac{T(r,z,t)}{qa/k} = \frac{1}{2}\int_0^\infty J_0(\lambda r)J_1(\lambda a) \times \left\{e^{-\lambda z}\operatorname{erfc}\left[\frac{z}{2(\alpha t)^{1/2}} - \lambda(\alpha t)^{1/2}\right] - e^{\lambda z}\operatorname{erfc}\left[\frac{z}{2(\alpha t)^{1/2}} + \lambda(\alpha t)^{1/2}\right]\right\}\frac{d\lambda}{\lambda}. \tag{2a}$$

This expression is valid for all r and z values equal to and greater than zero. At $z = 0$, the location of primary interest, this expression reduces to

$$\frac{T(r,0,t)}{qa/k} = \int_0^\infty \operatorname{erf}[\lambda(\alpha t)^{1/2}]\frac{J_0(\lambda r)J_1(\lambda a)}{\lambda}d\lambda \tag{2b}$$

which is difficult to evaluate numerically as mentioned above. For convenience, the above equations can be written in dimensionless form. Define

$$r^+ = \frac{r}{a}, z^+ = \frac{z}{a}, \lambda^+ = \lambda a, t^+ = \frac{\alpha t}{a^2} \tag{3a}$$

$$T^+(r^+, z^+, t^+) \equiv \frac{T(r^+, z^+, t^+)}{qa/k}. \tag{3b}$$

Then (2b) can be written (with the pluses dropped) as

$$T(r, 0, t) = \int_0^\infty \operatorname{erf}[\lambda t^{1/2}]\frac{J_0(\lambda r)J_1(\lambda)}{\lambda}d\lambda \tag{4}$$

and (2a) can be similarly written.

At the centerline ($r = 0$) Carslaw and Jaeger [6] present for any z the convenient dimensionless solution of

$$T(0, z, t) = 2t^{1/2}\left[\operatorname{ierfc}\left(\frac{z}{2t^{1/2}}\right) - \operatorname{ierfc}\left(\frac{(z^2+1)^{1/2}}{2t^{1/2}}\right)\right]. \tag{5}$$

For the heated surface ($z = 0$) and at $r = 0$, this equation yields

$$T(0, 0, t) = 2t^{1/2}\left[\frac{1}{\pi^{1/2}} - \operatorname{ierfc}\left(\frac{1}{2t^{1/2}}\right)\right] = \operatorname{erfc}\left(\frac{1}{2t^{1/2}}\right) + 2\left(\frac{t}{\pi}\right)^{1/2}[1 - e^{-1/4t}] \tag{6a}$$

which for small t values can be approximated by

$$T(0,0,t) \approx 2\left(\frac{t}{\pi}\right)^{1/2}[1 - 2t(1 - 6t + 60t^2)e^{-1/4t}]. \tag{6b}$$

For t values less than 0.02, and to 6 decimal places (6b) can be given simply by $2(t/\pi)^{1/2}$; this is the same expression as for the surface temperature of a semi-infinite body that is uniformly heated.

Thomas [7] derived an exact steady state solution for the surface temperature in terms of known functions. He gave for $0 < r < 1$,

$$T(r, 0, \infty) = \frac{2}{\pi}E(r) \tag{7}$$

and for $r > 1$

$$T(r, 0, \infty) = \frac{2r}{\pi}[E(r^{-1}) - (1 - r^{-2})K(r^{-1})]. \tag{8a}$$

At $r = 1$, $T(1, 0, \infty) = 2/\pi$. For 'large' r (8a) can be approximated by

$$T(r, 0, \infty) \approx \frac{1}{2r}\left[1 + \frac{1}{2(2r)^2} + \frac{3}{2^2(2r)^4} + \dots\right]. \tag{8b}$$

The leading term of (8b) is $1/2r$ which is the same as given by a point heat source. The functions $K(\cdot)$ and $E(\cdot)$ are the complete elliptic integrals of the first and second kinds,

$$K(\varepsilon) = \int_0^{\pi/2} [1 - \varepsilon^2 \sin^2 \theta]^{-1/2} \, d\theta \quad (9a)$$

$$E(\varepsilon) = \int_0^{\pi/2} [1 - \varepsilon^2 \sin^2 \theta]^{1/2} \, d\theta. \quad (9b)$$

These functions are tabulated in [9] and are available in computer libraries. This convenient exact solution for the steady state is utilized in the exact series solution developed below for the transient case.

SERIES SOLUTION

By using the relation between erfc(·) and erf(·), (4) can be given by

$$T(r, 0, t) = \int_0^{\infty} \frac{J_0(\lambda r)J_1(\lambda)}{\lambda} d\lambda - \int_0^{\infty} \operatorname{erfc}(\lambda t^{1/2}) \frac{J_0(\lambda r)J_1(\lambda)}{\lambda} d\lambda. \quad (10)$$

Notice that the first integral is a steady state term and the second term goes to zero as $t \to \infty$. Hence, the first integral is equal to either (7) or (8a) depending on the range of r.

Consider now the second integral in (10). The function erfc(·) has very small values for the argument greater than about 4. Hence, for evaluating the integral, only λ values less than $4t^{-1/2}$ need be considered. For 'large' t values this means that small λ's are of interest. Consequently, the defining series expressions for J_0 and J_1 can be effectively used; the first few terms of each are

$$J_0(\lambda r) = 1 - \frac{(\lambda r)^2}{4} + \frac{(\lambda r)^4}{4^2(2!)^2} - \frac{(\lambda r)^6}{4^3(3!)^2} + \ldots (11)$$

$$J_1(\lambda) = \frac{2\lambda}{4} - \frac{4\lambda^3}{4^2(2!)^2} + \frac{6\lambda^5}{4^3(3!)^2} - \ldots. \quad (12)$$

The product of these two functions when the λ coefficients are collected is

$$J_0(\lambda r)J_1(\lambda) = \sum_{i=0}^{\infty} (-1)^i (\lambda/2)^{2i+1} D_i(r) \quad (13)$$

where $D_i(r)$ is

$$D_i(r) = \sum_{j=1}^{i+1} (i-j+2) \left[\frac{r^{j-1}}{(j-1)!(i-j+2)!} \right]^2 \quad (14a)$$

where $i = 0, 1, 2, \ldots$. The first few values of $D_i(r)$ are

$$D_0(r) = 1, D_1(r) = \frac{1+2r^2}{2}, D_2(r) = \frac{1+6r^2+3r^4}{12} \quad (14b)$$

$$D_3(r) = \frac{1 + 12r^2 + 18r^4 + 4r^6}{144}. \quad (14c)$$

By using the integral relation for erfc(·), the second integral in (10) can be written as

$$I_2 = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \int_{\lambda t^{1/2}}^{\infty} e^{-u^2} du \frac{J_0(\lambda r)J_1(\lambda)}{\lambda} d\lambda$$

$$= \frac{2}{\sqrt{\pi}} \int_{t^{1/2}}^{\infty} \int_0^{\infty} e^{-\lambda^2 v^2} [J_0(\lambda r)J_1(\lambda)] d\lambda dv \quad (15)$$

where the substitution $u = \lambda v$ is used and the order of integration is interchanged. Note that

$$\int_0^{\infty} e^{-\lambda^2 v^2} \lambda^{2i+1} \, d\lambda = \frac{1}{2} \frac{i!}{2i + 2}. \quad (16)$$

Introducing (13) into (15), using (16) and then integrating over v produces the expression for (15) of

$$I_2 = \frac{1}{2\sqrt{\pi t}} \sum_{i=0}^{\infty} \frac{(-1)^i i!}{(2i + 1)(4t)^i} D_i(r). \quad (17)$$

A much more convenient form for computer (or programmable calculator) evaluation is

$$I_2 = -\frac{1}{2\sqrt{\pi t}} \sum_{k=1}^{\infty} \frac{(-1)^k}{C_{k-1}\, t^{k-1}} \sum_{j=1}^{k} \frac{k-j+1}{k} U_{kj}^2 \quad (18)$$

where

$$C_k = 4^k(2k + 1)[(k + 1)!] \quad (19)$$

$$U_{k1} = 1 \quad (20a)$$

$$U_{kj} = U_{k,j-1} \frac{(k-j+2)r}{j-1}; \quad k=1, 2, \ldots; \quad j=2, 3, \ldots, k \quad (20b)$$

where (20b) is a recursion relation.

In summary for $0 < r < 1$, a series expression for T at $z = 0$ is

$$T(r, 0, t) = \frac{2}{\pi} E(r) - I_2(r, t) \quad (21)$$

where the functional dependence of I_2 is noted. For $r > 1$, replace $(2/\pi)E(r)$ by (8a). The $I_2(r, t)$ function is calculated using (18)–(20). These exact expressions are very efficient for 'large' times because the infinite summation in I_2 can be approximated with just a few terms.

In order to display clearly the nature of the summation in I_2, several terms are now given,

$$T(r, 0, t) = T(r, 0, \infty) - \frac{1}{2\sqrt{\pi t}} \times \left\{ 1 - \frac{1+2r^2}{24t} + \frac{1}{480t^2}(1+6r^2+3r^4) - \frac{1}{10752t^3}(1+12r^2+18r^4+4r^6) + \ldots \right\}. \quad (22)$$

Note that the denominators 24, 480, etc. are the C_k values given by (19). A more extensive set of values C_k and of the coefficients of r^{2n} are given in Table 1.

Typical results for the local dimensionless temperature are given for various r and t values in Tables 2 and 3. The dimensionless times start at 0.04 and go to infinity. The number of terms required in the series increases quite rapidly as the dimensionless times become small, as shown in Table 4. Fortunately for a large range of t, the required number of terms is quite modest, e.g. less than 7 for $r = 0$ for $t > 1$ to obtain 8

Table 1. Coefficients of terms in equations (18) and (22)

k	C_k	Coefficients of r^{2n} $n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
0	1	1							
1	24	1	2						
2	480	1	6	3					
3	10 752	1	12	18	4				
4	276 480	1	20	60	40	5			
5	8 110 080	1	30	150	200	75	6		
6	268 369 920	1	42	315	700	525	126	7	
7	9 909 043 200	1	56	588	1960	2450	1176	196	8

Table 2. Values of local dimensionless temperatures for series solution given by equation (22). For radii equal to 0–1

Time, t	$r = 0.0$	$r = 0.25$	$r = 0.50$	$r = 0.75$	$r = 0.9$	$r = 1.0$
0.04	0.225647	0.225487	0.223512	0.208596	0.174404	0.106439
0.06	0.276003	0.275194	0.269833	0.245591	0.202785	0.128573
0.08	0.317551	0.315717	0.306483	0.274376	0.225293	0.146708
0.10	0.352882	0.349891	0.336874	0.298166	0.244201	0.162280
0.20	0.473895	0.466184	0.439395	0.379665	0.311190	0.219495
0.40	0.595213	0.583412	0.545075	0.468351	0.388229	0.288571
0.60	0.659147	0.645795	0.603023	0.519584	0.434689	0.331665
0.80	0.700063	0.685949	0.640975	0.554129	0.466752	0.361944
1.00	0.729097	0.714546	0.668298	0.579454	0.490599	0.384714
2.00	0.804583	0.789251	0.740710	0.648203	0.556583	0.448652
4.00	0.860404	0.844767	0.795324	0.701345	0.608589	0.499803
10.00	0.911164	0.895396	0.845562	0.750939	0.657679	0.548510
40.00	0.955443	0.939636	0.889682	0.794861	0.701444	0.592155
100.00	0.971802	0.955991	0.906024	0.811181	0.717747	0.608445
400.00	0.985897	0.970084	0.920113	0.825264	0.731825	0.622519
1000.00	0.991080	0.975267	0.925295	0.830446	0.737006	0.627700
4000.00	0.995540	0.979727	0.929755	0.834905	0.741465	0.632160
10 000.00	0.997179	0.981366	0.931395	0.836544	0.743105	0.633799
40 000.00	0.998590	0.982776	0.932805	0.837955	0.744515	0.635209
100 000.00	0.999108	0.983295	0.933323	0.838473	0.745033	0.635728
Infinity	1.000000	0.984187	0.934215	0.839365	0.745926	0.636620

Table 3. Values of local dimensionless temperatures for series solution given by equation (22). For radii equal to 1.25–8

Time, t	$r = 1.25$	$r = 1.50$	$r = 2.00$	$r = 4.00$	$r = 8.0$
0.08	0.029169				
0.10	0.037316	0.009490			
0.20	0.072147	0.027664	0.003456		
0.40	0.121933	0.060794	0.014967		
0.60	0.156405	0.087005	0.027959		
0.80	0.181959	0.107766	0.040180	0.000423	
1.00	0.201817	0.124566	0.051148	0.001005	
4.00	0.308779	0.222454	0.129605	0.021057	0.000340
10.00	0.356370	0.268718	0.172666	0.047613	0.004760
40.00	0.399662	0.311585	0.214469	0.082881	0.023344
100.00	0.415914	0.327789	0.230554	0.098174	0.035858
400.00	0.429976	0.341837	0.244566	0.111944	0.048705
1000.00	0.435156	0.347015	0.249741	0.117092	0.053750
4000.00	0.439615	0.351474	0.254198	0.121541	0.058168
10 000.00	0.441254	0.353113	0.255837	0.123180	0.059803
40 000.00	0.442664	0.354523	0.257247	0.124590	0.061213
100 000.00	0.443183	0.355042	0.257766	0.125108	0.061731
Infinity	0.444075	0.355934	0.258658	0.126000	0.062623

Table 4. Number of terms on series of equation (21) to obtain 3 and 8 significant figure accuracy

Time	$r = 0$		$r = 1$		$r = 2$	
	3 sign. f.	8 sign. f.	3 sign. f.	8 sign. f.	3 sign. f.	8 sign. f.
0.01	65	<75				
0.05			49	60		
0.1	9	16	25	34	63	75
0.2	5	11				
1	2	7	4	9	8	14
10			2	5	2	6
100					2	3

significant figure accuracy. Also the number of additional terms required to go from 3 to 8 significant figures is not large. The series solution, however, is not appropriate for very small dimensionless times. The limiting appropriate dimensionless times are about 0.01, 0.05 and 0.1 for $r = 0$, 1 and 2, respectively. For $r \geq 1$ a convenient limiting time expression is

$$t/r^2 \geq 0.05. \tag{23}$$

Below this time the number of terms becomes ever larger and the series does not always converge to the correct value using the CDC 6600 with 14 significant figure accuracy. Double precision does not seem to help significantly. Since t/r^2 is smaller than 0.05 for some of the possible entries in Table 3 and convergence may not have been obtained, some values on the upper right are omitted.

Temperatures for $r = 0$, 0.25, 0.5, 0.75, 0.9 and 1.0 are given in Table 2 and are plotted in Fig. 2. For the small t values at $r = 0$, temperatures were calculated utilizing (6a). The $r = 1$ curve for small t was found using

$$T(1, 0, t) \approx \left(\frac{t}{\pi}\right)^{1/2} - \frac{t}{2\pi}\left(1 + \frac{t}{8} + \frac{9t^2}{96}\right) \tag{24}$$

which is accurate to 5 significant figures for $t < 0.1$. This expression was derived using a quite different procedure and so will not be discussed further here. For very small t values (about 10^{-4}) the T given by (24) is one-half the center value given by (6).

Except for the r values equal to and less than 1.25,

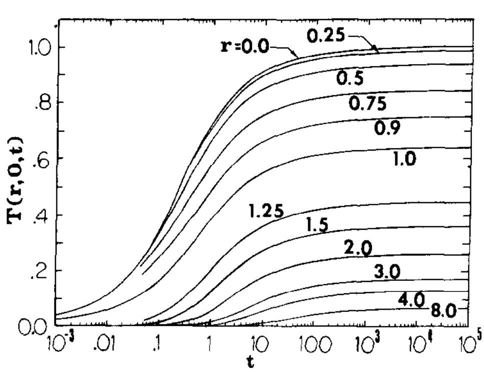

FIG. 2. Local temperature vs time at $z = 0$.

Tables 2 and 3 provide values of T that give a fairly complete set of curves as shown by Fig. 2. The temperatures for $r < 1$ respond immediately to the onset of heating but there is a lag for $r > 1$ that increases with r.

For large radii the disk heat source behaves as if it were a point source which has a solution of

$$T(r, t) = \frac{1}{2r}\operatorname{erfc}[r/(2t^{1/2})]. \tag{25}$$

For steady state (25) gives for $r = 8$ the value of 0.0625 while the Table 3 value is 0.062623 which is 0.2% higher. For larger r the error in using (25) is less but the percent error for a given r tends to become larger as t is reduced.

AVERAGE TEMPERATURE

The temperature averaged over position is of interest for determining the contact conductance and other purposes. For the average temperature between $r = 0$ and $r = c$ (a dimensionless arbitrary radial location) one can multiply $T(r, 0, t)$ by $2\pi r dr$, integrate from $r = 0$ to c and divide by πc^2. The result is

$$\bar{T}(c, 0, t) = \bar{T}(c, 0, \infty) - \bar{I}_2(c, t) \tag{26}$$

where $\bar{I}_2(c, t)$ is exactly the same expression as given by (18) except the inner summation has kj in the denominator instead of simply k and in (20b), r is replaced by c. The term $\bar{T}(c, 0, \infty)$ in (26) for $0 < c \leq 1$ is given by (see [11] for annular heating)

$$\bar{T}(c, 0, \infty) = \frac{4}{3\pi c^2}[(1 + c^2)E(c) - (1 - c^2)K(c)] \tag{27a}$$

and for $1 \leq c < \infty$

$$\bar{T}(c, 0, \infty) = \frac{4}{3\pi c}[(1+c^2)E(c^{-1}) + (1 - c^2)K(c^{-1})]. \tag{27b}$$

At $c = 0$, $\bar{T}(0, 0, \infty) = 1$ and at $c = 1$, $\bar{T}(1, 0, \infty) = 8/3\pi$. For large c values (27b) can be approximated by

$$\bar{T}(c, 0, \infty) \approx \frac{1}{c}\left[1 - \frac{1}{8c^2} - \frac{1}{64c^4}\right]. \tag{27c}$$

An expanded form of (26) for a few terms is

$$\bar{T}(c,0,t) = \bar{T}(c,0,\infty) - \frac{1}{2\sqrt{\pi t}} \times \left[1 - \frac{1+c^2}{24t} + \frac{1}{480t^2}(1+3c^2+c^4) - \frac{1}{10752t^3}(1+6c^2+6c^4+c^6) + \ldots\right] \tag{28}$$

The coefficients of c^{2n} are given in Table 5 in a similar manner as Table 1 gives the coefficients for (18). Unlike the coefficients given in Table 1, those in Table 5 display a symmetry for a given k value. The sum given in Table 5 is the sum of the coefficients for a given k value and can be used for $c = 1$.

Tables 6 and 7 provide values for $\bar{T}(r, 0, t)$ for specific r and t. Fewer terms in the series given by (28) are needed than are given in Table 4. The $r = 0$ average value is the same as the local value. For small t the average temperature from $r = 0$ to 1 can be approximated by [8]

$$\bar{T}(1,0,t) \approx 2\left(\frac{t}{\pi}\right)^{1/2} - \frac{t}{\pi}\left[2 - \frac{t}{4} - \left(\frac{t}{4}\right)^2 - \frac{15}{4}\left(\frac{t}{4}\right)^3\right] \tag{29}$$

which is accurate to 5 significant digits for $0 < t < 0.1$. Again in Table 7 as in Table 3 certain $\bar{T}$ values on the upper right are omitted due to possible non-convergence.

The average temperatures of Tables 6 and 7 are plotted in Fig. 3. The curve of $\bar{T}$ for small t and for $r > 1$ shown in Fig. 3 can be obtained by using

$$\bar{T}(c,0,t) \approx \frac{2}{c^2}\left(\frac{t}{\pi}\right)^{1/2}. \tag{30}$$

This expression becomes more accurate as $t \to 0$ and as c becomes larger. For $c = 1.5$ and $t = 0.2$ it gives a number that is 5% too large but for $c = 8$ and $t = 4$ the value given by (30) is only 0.2% large.

A comparison of Figs 2 and 3 shows that they have the same general shape but the average curves start to

Table 5. Coefficients of c^{2n} for the average temperature expression given by equation (28)

k	Sum	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
0	1	1							
1	2	1	1						
2	5	1	3	1					
3	14	1	6	6	1				
4	42	1	10	20	10	1			
5	132	1	15	50	50	15	1		
6	429	1	21	105	175	105	21	1	
7	1430	1	28	196	490	490	196	28	1

Table 6. Values of average dimensionless temperatures for series solution given by equation (26). For radii equal to 0–1

Time, t	$c=0.0$	$c=0.25$	$c=0.50$	$c=0.75$	$c=0.9$	$c=1.0$
0.04	0.225647	0.225579	0.224952	0.220961	0.212710	0.200340
0.06	0.276003	0.275631	0.273607	0.265885	0.253896	0.238489
0.08	0.317551	0.316679	0.312874	0.301553	0.286491	0.268745
0.10	0.352882	0.351438	0.345798	0.331232	0.313611	0.293981
0.20	0.473895	0.470087	0.457473	0.431853	0.406122	0.380695
0.40	0.595213	0.589350	0.570812	0.536092	0.503790	0.473675
0.60	0.659147	0.652506	0.631706	0.593446	0.558534	0.526521
0.80	0.700063	0.693039	0.671121	0.631088	0.594845	0.561854
1.00	0.729097	0.721854	0.699291	0.658225	0.621200	0.587630
2.00	0.804583	0.796949	0.773229	0.730289	0.691826	0.657178
4.00	0.860404	0.852617	0.828444	0.784759	0.745716	0.710628
10.00	0.911164	0.903312	0.878943	0.834933	0.795636	0.760354
40.00	0.955443	0.947571	0.923142	0.879034	0.839658	0.804316
100.00	0.971802	0.963928	0.939493	0.895373	0.855989	0.820640
400.00	0.985897	0.978022	0.953585	0.909462	0.870075	0.834725
1000.00	0.991080	0.983205	0.958767	0.914644	0.875257	0.839906
4000.00	0.995540	0.987665	0.963227	0.919104	0.879717	0.844366
10 000.00	0.997179	0.989304	0.964867	0.920743	0.881356	0.846005
40 000.00	0.998590	0.990715	0.966277	0.922154	0.882767	0.847416
100 000.00	0.999108	0.991233	0.966795	0.922672	0.883285	0.847934
Infinity	1.000000	0.992125	0.967688	0.923564	0.884177	0.848826

Table 7. Values of average dimensionless temperatures for series solutions given by equation (26). For radii equal to 1.25–8

Time, t	$c = 1.25$	$c = 1.50$	$c = 2.00$	$c = 4.00$	$c = 8.0$
0.10	0.216281				
0.20	0.288068	0.214035	0.125371		
0.40	0.368792	0.282596	0.172899		
0.60	0.416518	0.324954	0.204969		
0.80	0.449158	0.354670	0.228736		
1.00	0.473315	0.377034	0.247300	0.070295	
2.00	0.539805	0.440036	0.302517	0.097076	
4.00	0.591984	0.490716	0.349646	0.127344	0.035189
6.00	0.616242	0.514565	0.372493	0.144682	0.042764
8.00	0.630979	0.529125	0.386615	0.156231	0.048742
10.00	0.641141	0.539191	0.396444	0.164621	0.053613
40.00	0.684925	0.582760	0.439471	0.204201	0.082941
100.00	0.701230	0.599042	0.455693	0.220020	0.097285
400.00	0.715309	0.613113	0.469746	0.233951	0.110745
1000.00	0.720490	0.618294	0.474925	0.239117	0.115859
4000.00	0.724950	0.622753	0.479384	0.243572	0.120298
10 000.00	0.726589	0.624392	0.481023	0.245211	0.121935
40 000.00	0.727999	0.625802	0.482433	0.246621	0.123345
100 000.00	0.728518	0.626321	0.482952	0.247139	0.123863
Infinity	0.729410	0.627213	0.483843	0.248031	0.124755

rise sooner and reach larger steady state values. This is true for all curves except for $c = r = 0$ for which the curves are identical.

CASE OF z GREATER THAN ZERO

Some of the very important regions of $z > 0$ can be treated utilizing a similar procedure to that given above for $z = 0$. A convenient starting expression is

$$T(r, z, t) = \int_0^t (\pi\theta)^{-1/2} e^{-z^2/4\theta} \times \int_0^\infty e^{-\theta\lambda^2} J_0(\lambda r) J_1(\lambda) d\lambda d\theta. \quad (31)$$

This dimensionless expression can be derived from equation (9) of p. 260 of [6]. By using (13) for the product of the Bessel functions and employing the typical integral of

$$\int_0^\infty e^{-\theta\lambda^2} \lambda^{2i+1} d\lambda = \frac{i!}{2\theta^{i+1}} \quad (32)$$

one can derive

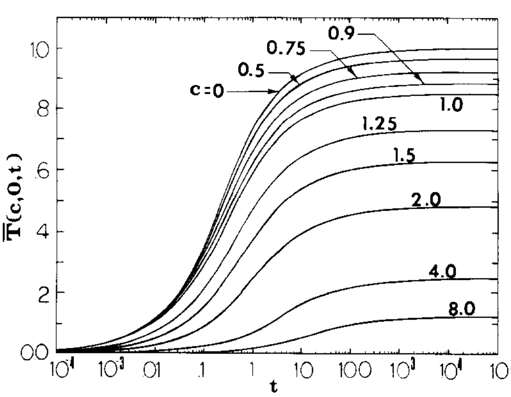

FIG. 3. Average temperature vs time at $z = 0$.

$$T(r, z, t) = \frac{1}{4}\int_0^t (\pi\theta)^{-1/2} e^{-z^2/4\theta} \times \sum_{i=0}^{\infty} \frac{(-1)^i i!}{(4\theta)^i} D_i(r) d\theta. \quad (33)$$

A typical integral of (33) is

$$\int_0^t e^{-z^2/4\theta} \frac{d\theta}{\theta^{i+3/2}} = \left(\frac{2}{z}\right)^{2i+1} \Gamma\left(i + \frac{1}{2}, \frac{z^2}{4t}\right) \quad (34)$$

where z is not permitted to be zero. As a consequence, this procedure cannot be used for $z = 0$. Now using (34) in (33) produces

$$T(r, z, t) = \frac{1}{2\pi^{1/2}} \sum_{i=0}^{\infty} \frac{(-1)^i i!}{z^{2i+1}} \times D_i(r) \Gamma\left(i + \frac{1}{2}, \frac{z^2}{4t}\right) \quad (35)$$

the first few terms of which are

$$T(r, z, t) = \frac{1}{2z\pi^{1/2}} \left[\Gamma\left(\frac{1}{2}, \frac{z^2}{4t}\right) - \frac{1 + 2r^2}{2z^2} \Gamma\left(\frac{3}{2}, \frac{z^2}{4t}\right) + \frac{1 + 6r^2 + 3r^4}{6z^4} \Gamma\left(\frac{5}{2}, \frac{z^2}{4t}\right) - \dots \right] \quad (36)$$

By taking advantage of some relations for the incomplete gamma function, $\Gamma(i + 1/2, x^2)$, the T expression given by (35) can be written in terms of more familiar functions. Let

$$\Gamma(i + 1/2, x^2) = \Gamma(i + 1/2)[1 - V_{i-1}(x)] \quad (37)$$

where $V_{i-1}(x)$ is defined herein to make (37) true. It is $V_{-1}(x) \equiv \mathrm{erf}(x)$ and

$$V_{i-1}(x) \equiv \sum_{j=i}^{\infty} W_j(x), \; i = 0, 1, 2, \dots \quad (38a)$$

where $W_j(x)$ is defined to be

Table 8. $V_{i-1}(x)$ values for x and i values

$i-1$	$x=0.1$	$x=0.5$	$x=1$	$x=2$	$x=4$
0		0.081110			1.00000
1	0.29876E-5	0.78767E-2	0.15085	0.84376	0.99999
2	0.85306E-8	0.55352E-3	0.04016	0.66741	0.99996
3	0.1895E-10	0.30434E-4	0.00853	0.46585	0.99980
4		0.13735E-5	0.00150	0.28670	0.99924
5		0.52549E-7	0.00023	0.15640	0.99760
6		0.17446E-8	0.00003	0.07622	0.99356
10			0.29E-8	0.00163	0.89986
20				0.8E-9	0.10895
30					0.00040
40					0.77E-7

$$W_j(x) \equiv \frac{4}{\sqrt{\pi}} x e^{-x^2} \frac{(j+1)!(4x^2)^j}{(2j+2)!} \tag{38b}$$

for $j = 0, 1, 2 \ldots$ and $W_{-1}(x) \equiv 0$.

The recursion relation

$$W_{j+1}(x) = \frac{x^2}{j+1.5} W_j(x), \; j = 0, 1, 2, 3 \ldots \tag{38c}$$

can be used to provide an efficient method of evaluation of the terms in (38a). Notice that $V_{i-1}(x)$ in (37) goes to zero for any x provided i is sufficiently large (see (38a) and (38c)). This results in reducing the number of terms needed for transient solutions given by (35).

The gamma function $\Gamma(i + 1/2)$ can be written as

$$\Gamma(i + 1/2) = \frac{(2i)!}{4^i(i!)}\sqrt{\pi}. \tag{39}$$

Then introducing (37) and (39) into (35) yields

$$T(r, z, t) = \frac{1}{2z}\sum_{i=0}^{\infty}\frac{(-1)^i(2i)!}{(2z)^{2i}} \times D_i(r)\left[1 - V_{i-1}\left(\frac{z}{2t^{1/2}}\right)\right]. \tag{40}$$

Notice that (40) provides a steady state and a transient part. The former is

$$T(r, z, \infty) = \frac{1}{2z}\sum_{i=0}^{\infty}\frac{(-1)^i(2i)!}{(2z)^{2i}} D_i(r) \tag{41a}$$

$$= \frac{1}{2z}\left[1 - \frac{1+2r^2}{(2z)^2} + \frac{2(1+6r^2+3r^4)}{(2z)^4} - \frac{5(1+12r^2+18r^4+4r^6)}{(2z)^6} + \ldots\right]. \tag{41b}$$

This expression yields accurate values for T provided

$$2z \gg r \text{ and } 2z \gg 1$$

which means in actual practice that $z > 2.5$ for $0 < r < 1$ and $z > 2.5r$ for $r > 1$.

The number of transient terms caused by $V_{i-1}(z/2t^{1/2})$ is not as large as one might expect, particularly for small values of $z/2t^{1/2}$ (see Table 8). Not only do the coefficients of $V_{i-1}(\cdot)$ decrease as shown by (41b) but $V_{i-1}(\cdot)$ always goes to zero with increasing i values, i.e. increasing number of terms for the summation in (40). For example, for $z/2t^{1/2}$ less than 0.5 and 2 the number of terms is respectively less than 6 and 20 to obtain 8 significant figures in the transient term.

OTHER CASES

The geometry discussed herein is a basic building block that can be utilized to provide the solutions for many related geometries. These include various heating conditions, geometries and boundary conditions. Beck [8] discusses a number of these but even more are possible for the local temperature distribution than for the average that is discussed in [8]. For example, annular heating from $r = a_1$ to a_2 can be treated by subtracting the solution for $a = a_2$ from that obtained for $a = a_1$. (Care must be taken to include all the dependences on a_1 and a_2.) For an arbitrary heat flux $q(r)$ the local temperature rise is

$$T(r, 0, t) = \int_0^{r_{max}} q(r')\frac{\partial\phi(r, a = r', z, t)}{\partial a}\,dr' \tag{42}$$

where the maximum radius of non-zero $q(r)$ is r_{max} and where r' is a dummy variable. In (42) each quantity has units, that is, it is not in dimensionless form. The dimensional temperature $T(r, z, t)$ for a constant heat flux q of unity is denoted $\phi(r, a, z, t)$.

A further use of the solution contained herein is for the transient contact conductance for a regular distribution of contacts. This can be accomplished by extending the results of [10].

CONCLUSIONS

A new series solution is given for the transient temperatures in a semi-infinite solid heated over a circular area. The solution at the surface takes advantage of a known steady state solution. By concentrating on 'large' times it is found possible to obtain results even down to the small dimensionless times of 0.01. This procedure utilizing the steady state may be effective in other related problems.

The problem is a basic one and the solution can be used as a building block in a number of other geometries and boundary conditions, some of which are mentioned. It is also a fundamental solution for use in connection with a new solution method called the surface element method. This method is competitive with the finite difference and element methods when unlike geometries are attached such as a rod and a semi-infinite solid.

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SOLUTION, POUR LES GRANDES ECHELLES DE TEMPS, DE LA TEMPERATURE DANS UN MILIEU SEMI-INFINI AVEC UNE SOURCE DE CHALEUR CIRCULAIRE

Résumé—On présente une solution série pour l'histoire de la température locale de surface d'un milieu semi-infini chauffé seulement sur une aire circulaire. Dans cette région le flux thermique est constant dans le temps et uniforme, tandis qu'en dehors de cette aire la surface est isolée. Un certain nombre de solutions approchées est donné dans la littérature. Une solution exacte est connue mais elle se présente sous la forme d'une intégrale avec un domaine infini. La solution développée ici est plus pratique pour tous les temps sans dimension à l'exception des plus petits. On fournit des courbes et des tables.

En plus de la solution pour la surface, on donne une solution pour certains points à l'intérieur mais aussi pour les époques "lointaines".

La solution est importante parce qu'elle est fondamentale dans la conduction thermique et elle est fréquemment utilisée pour les corps cylindriques. La solution peut être employée pour les geométries finies, pour des chauffages variables dans le temps et dans le cas de flux thermiques variant spatialement de façon symétrique. Elle peut aussi être utilisée dans une nouvelle méthode prometteuse et qui est appellée la méthode des éléments de surface.

LANGZEITLÖSUNGEN FÜR DIE TEMPERATUR IN EINEM HALBUNENDLICHEN KÖRPER MIT EINER WÄRMEQUELLE IN FORM EINER KREISFLÄCHE

Zusammenfassung — Die Arbeit gibt eine Lösung für den örtlichen und zeitlichen Temperaturverlauf an der Oberfläche eines halbunendlichen Körpers an, der ausschließlich über ein kreisförmiges Gebiet beheizt wird. Innerhalb dieses Gebiets ist der Wärmestrom zeitlich und örtlich konstant, während die Oberfläche außerhalb der Kreisfläche isoliert ist. Aus der Literatur sind hierfür eine Anzahl von Näherungslösungen bekannt. Es existiert auch eine exakte Lösung, aber sie hat die Form eines unendlichen Integrals. Die hier entwickelte Lösung ist für alle dimensionslosen Zeiten — ausgenommen die kleinsten — sehr viel bequemer anzuwenden. Ausführliche Diagramme und Tabellen werden angegeben.

Zusätzlich zur Lösung für die Oberfläche gibt es eine Lösung für bestimmte Orte innerhalb des Körpers ebenfalls für "große" Zeiten.

Die Lösung ist wichtig, weil sie eine grundlegende Geometrie der Wärmeleitung betrifft und häufig im Zusammenhang mit zylindrischen Körpern gebraucht wird. Sie kann als Baustein für verwandte finite Geometrieen bei zeitabhängiger Erwärmung und symmetrischen räumlich veränderlichen Wärmeströmen benutzt werden. Die Lösung kann ebenfalls innerhalb einer vielversprechenden neuen Berechnungsmethode, der sogenannten Oberflächen-Element-Methode, verwendet werden.

ТЕМПЕРАТУРНОЕ ПОЛЕ ПРИ БОЛЬШИХ ВРЕМЕНАХ В ПОЛУБЕСКОНЕЧНОМ ТЕЛЕ С ИСТОЧНИКОМ ТЕПЛА В ФОРМЕ ДИСКА

Аннотация – Представлено решение в виде ряда для локального изменения во времени температуры поверхности полубесконечного тела, нагреваемого по круговой области. Внутри этой области тепловой поток является стационарным и однородным, в то время как снаружи поверхность изолирована. Для данного случая в литературе имеется ряд приближенных решений и одно точное решение, но в виде интеграла с бесконечной областью. Предлагаемое в работе решение является гораздо более удобным, когда используются безразмерные времена за исключением наименьших. Приведены также многочисленные кривые и таблицы.

Помимо решения для поверхности дано решение для некоторых областей внутри полубесконечного тела и тоже для «больших» значений времени.

Предлагаемое решение имеет важное значение в связи с тем, что рассматриваемая геометрия является основной в задачах теплопроводности тел цилиндрической формы. Оно может использоваться как составная часть при решении задач для конечных геометрий при нестационарном нагреве и при рассмотрении симметричных случаев с неравномерным тепловым потоком. Кроме того, его можно использовать в новом перспективном методе расчета, называемом методом элементарной площадки.